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Frames, Young Tableaux, and Baxter Sequences

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1. INTRODUCTION

From a partition of a number, it is possible to construct a diagram which is variously known as a *Ferrers' diagram*, a *Young diagram*, a *partition diagram*, a *frame of a partition*, etc. This diagram is usually drawn as a pattern of squares or "boxes" in the plane. Given a frame of a partition, one then can place a positive integer in each of these squares, subject to the restrictions that these numbers must be nondecreasing along each row of squares and strictly increasing down each column, to create what is frequently called a *Young tableau*, although this terminology is by no means universal. There is a well-known relationship between Young tableaux and the monomial expression of Schur functions (see [3, p. 191]).

Given a partition of a number n, certain special cases of Young tableaux are of particular interest. These are the *standard tableaux*. A standard tableau is a Young tableau in which the numbers $1, 2, \ldots, n$ each occur exactly once. The purpose of this paper is to show first, that the set of all Young tableaux can be divided into equivalence classes each of which is characterized by a standard tableau; and second, to describe an operator by which a set of monomials corresponding to the Young tableaux of an equivalence class can be generated. It is this operator which I term a *Baxter sequence*.

A Baxter sequence is formed from a pair of *Baxter operators*. (Baxter operators are a class of operators introduced by Baxter [1] in fluctuation theory.) One of these operators was described and much used by Rota in [5]. The other operator appears in Rota and Smith's work on fluctuation theory [8].

A Baxter sequence is constructed from a standard tableau and is closely related to other sequences constructed from standard tableaux by Foulkes [2] under the name *line of route*, and Schützenberger [6]. A consequence of this is that for some purposes previously involving the use and manipulation of Young tableaux, consideration of standard tableaux is now all that is required.

The starting point for this work was an investigation into the effects of changing the number of variables or indeterminates in which Schur functions could be expressed. The use of Baxter sequences solves this problem since it allows one

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to deal with Schur functions expressed in terms of 1, 2, 3,... indeterminates simultaneously.

The results of this paper are expressed in terms which are more general than Young tableaux. Taking the special case of Young tableaux alone does not affect the results in any way, and in fact, the proof of Theorem 2 is considerably easier in the special case.

2. Frames and Numberings

2.1. Definitions. Let Z denote the set of integers. A *frame* is defined as any finite subset F of $Z \times Z$.

A *numbering* of a frame F is a map η from F to Z^+ (the set of positive integers) satisfying the two conditions;

(i) $\eta(i, j) \leqslant \eta(i', j')$ whenever $i = i'$ and $j < j'$,

(ii) $\eta(i, j) < \eta(i', j')$ whenever $j = j'$ and $i < i'$.

A *standard numbering* of F is a numbering ζ of F such that ζ is a one-to-one map of F onto the set $\{1, 2, \ldots, n\}$ where $n = |F|$.

2.2. Example. We shall adopt the following conventions for writing F and η. Suppose $F = \{(1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$. Then we shall express F as a pattern of squares:

(1, 1)	(1, 2)	
	(2, 2)	
(3, 1)		(3, 3)

Note the direction of the axes. The second index increases toward the right, and the first index increases down the page.

A numbering η of F can now be expressed by placing the integer $\eta(i, j)$ in the square (i, j).

For example, a numbering could be expressed

1	2	
	5	
2		2

Clearly, if F is a frame of a partition, then a numbering η of F produces a Young tableau.

2.3. *Index Numberings*

We shall denote by D_F the set of all numberings of a frame F.

Consider a numbering $\eta \in D_F$. We can construct a total ordering of the squares of F as follows.

If $(i, j), (i', j') \in F$, then (i, j) precedes (i', j') if either

$$\eta(i, j) < \eta(i', j'),$$

or

$$\eta(i, j) = \eta(i', j') \qquad \text{and } i > i',$$

or

$$\eta(i, j) = \eta(i', j'), \qquad i = i' \text{ and } j < j'.$$

Hence, corresponding to η, suppose the squares of F are ordered

$$(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n).$$

We can now define a standard numbering ζ of F by putting

$$\zeta(i_r, j_r) = r \qquad \text{for} \quad r = 1, 2, \ldots, n.$$

A standard numbering ζ thus constructed from a numbering η will be called the *index numbering* of F corresponding to η. For example, from

$$\begin{array}{ccc} \boxed{1} & \boxed{2} & \\ & \boxed{4} & \\ \boxed{2} & & \boxed{4} \end{array}$$

construct the index numbering

$$\begin{array}{ccc} \boxed{1} & \boxed{3} & \\ & \boxed{5} & \\ \boxed{2} & & \boxed{4} \end{array}$$

2.4. *An Equivalence Relation on D_F*

We can now define an equivalence relation on D_F by saying that two members of D_F are equivalent if they have the same corresponding index numbering.

Hence, D_F can be split into a finite number of equivalence classes each of which can be characterized by a standard numbering of F.

3. The Inventory of a Frame

3.1. *Monomials*

Consider a numbering $\eta \in D_F$. Let $p(r) = |\{(i,j) \in F \text{ such that } \eta(i,j) = r\}|$ for $r = 1, 2, \ldots$ (i.e., $p(r)$ is the number of squares which are numbered with an r).

Define a monomial $M(\eta)$ corresponding to η by

$$M(\eta) = x_1^{p(1)} x_2^{p(2)} x_3^{p(3)} \ldots = \prod_{r=1}^{\infty} x_r^{p(r)},$$

where $x_1, x_2, x_3, \ldots$ is an infinite set of variables or indeterminates.

3.2. Lemma 1. *A numbering $\eta \in D_F$ of a frame F is uniquely determined by index numbering of F corresponding to η taken together with $M(\eta)$.*

Proof. Suppose $F = \{(i_r, j_r)\}_{r=1}^{n}$ and the index numbering is ζ such that

$$\zeta(i_r, j_r) = r \qquad \text{for} \quad r = 1, 2, \ldots, n.$$

Suppose $M(\eta) = x_{e(1)} x_{e(2)} \ldots x_{e(n)}$ where $e(1) \leqslant e(2) \leqslant \cdots \leqslant e(n)$. Then we can derive η by putting

$$\eta(i_r, j_r) = e(r) \qquad \text{for} \quad r = 1, 2, \ldots, n.$$

Corollary. *If η and $\eta' \in D_F$ $(\eta \neq \eta')$ and $M(\eta) = M(\eta')$ then η and η' cannot belong to the same equivalence class.*

3.3. *Definition of the Inventory*

Let $C_F \subseteq D_F$ be a subset of D_F. Define the *inventory* of C_F by

$$\sum_{\eta \in C_F} M(\eta)$$

where the summation is over all $\eta \in C_F$.

Denote the inventory of C_F by $I(C_F)$.

In Section 5 we shall be concerned with the determination of the inventories of the equivalence classes of D_F.

$I(D_F)$ is not always easy to deal with. Consequently, we make the following definitions.

Define $C_F{}^r = \{\eta \in C_F \mid p(r) > 0, \ p(r') = 0 \text{ for all } r' > r\}$ and $I^r(C_F) = \sum_{\eta \in C_F{}^r} M(\eta)$.

Now define $\bar{I}(C_F)$ to be the infinite sequence

$$(I^1(C_F), I^2(C_F), I^3(C_F), \ldots).$$

Note that $I(C_F) = \sum_{r=1}^{\infty} I^r(C_F)$.

$\bar{I}(C_F)$ will generally be easier to determine than $I(C_F)$.

From now on the word *inventory* may refer to either $I(C_F)$ or $\bar{I}(C_F)$ but it should be clear from the context to which one we refer.

4. Baxter Sequences

4.1. *Baxter Operators*

Let A be a commutative algebra over a field K of characteristic zero. A *Baxter operator* on A is a linear operator $B\colon A \to A$ such that for some fixed $\theta \neq 0$ in K;

$$B(aB(b)) + B(bB(a)) = B(a)\,B(b) + B(\theta ab)$$

for all $a, b \in A$.

The pair (A, B) will be called a *Baxter algebra.*

4.2. Example. The following examples of Baxter operators may be found in Rota and Smith [8]. Let K be a field of characteristic zero and let A be the algebra of infinite sequences $(a_1, a_2, \ldots)$ with entries in K considered as an algebra over K in which all operations are componentwise.

Define $S\colon A \to A$ by

$$S(a_1, a_2, a_3, \ldots, a_r, \ldots) = \left(0, a_1, a_1 + a_2, \ldots, \sum_{i=1}^{r-1} a_i, \ldots\right).$$

Rota [5] showed that S is a Baxter operator on A for $\theta = -1$.

We can define another Baxter operator P on A by

$$P(a_1, a_2, \ldots, a_r, \ldots) = \left(a_1, a_1 + a_2, \ldots, \sum_{i=1}^{r} a_i, \ldots\right).$$

This can be shown to be a Baxter operator on A for $\theta = +1$.

Henceforth, $(A; P, S)$ will be taken to be the Baxter algebra as defined in this paragraph.

In addition, define $x \in A$ to be $(x_1, x_2, x_3, \ldots)$ where $x_1, x_2, x_3, \ldots$ is an infinite sequence of indeterminates in K.

4.3. *Baxter Sequences*

A *Baxter sequence element* is an element of $(A; P, S)$ of the form

$$xB_{n-1}(xB_{n-2}(xB_{n-3}(\ldots (xB_1(x)) \ldots)))$$

where each B_i is either the P or S Baxter operator.

This expression will usually be abbreviated to

$$B_1B_2 \cdots B_{n-1}(x).$$

(Note the reversal of order of the B_i's).

Note also that $B_1B_2 \cdots B_{n-1}(x) = xB_{n-1}(B_1B_2 \cdots B_{n-2}(x))$.

A *Baxter sequence* is defined as a sequence of symbols $B_1B_2 \cdots B_{n-1}$ where each B_i is either the symbol P or S.

4.4. Example.

$$\begin{aligned}
x &= (x_1, x_2, \ldots),\\
P(x) &= (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots),\\
xP(x) &= (x_1^2, x_1x_2 + x_2^2, x_1x_3 + x_2x_3 + x_3^2, \ldots),\\
S(xP(x)) &= (0, x_1^2, x_1^2 + x_1x_2 + x_2^2, x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3, \ldots),\\
xS(xP(x)) &= (0, x_1^2x_2, x_1^2x_3 + x_2^2x_3 + x_1x_2x_3,\\
&\qquad x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \ldots).
\end{aligned}$$

We may rewrite this example simply noting the rth entries:

$$x = (\ldots, x_r, \ldots);$$

$$P(x) = \left(\ldots, \sum_{i=1}^{r} x_i, \ldots\right);$$

$$xP(x) = \left(\ldots, \sum_{i=1}^{r} x_ix_r, \ldots\right);$$

$$S(xP(x)) = \left(\ldots, \sum_{1\leqslant i\leqslant j<r} x_ix_j, \ldots\right);$$

$$xS(xP(x)) = \left(\ldots, \sum_{1\leqslant i\leqslant j<r} x_ix_jx_r, \ldots\right).$$

We could continue with

$$S(xS(xP(x))) = \left(\ldots, \sum_{1\leqslant i\leqslant j<k<r} x_ix_jx_k, \ldots\right),$$

$$xS(xS(xP(x))) = \left(\ldots, \sum_{1\leqslant i\leqslant j<k<r} x_ix_jx_kx_r, \ldots\right).$$

Clearly, for any Baxter sequence, $B_1B_2 \cdots B_{n-1}(x)$ is an infinite sequence in which each term is a sum of monomials of degree n.

4.5. *Some Lemmas on Baxter Sequences*

LEMMA 2. *Suppose $x_{e(1)}x_{e(2)} \dots x_{e(n)}$ is a monomial appearing in the* rth *term of the Baxter sequence element $B_1 B_2 \cdots B_{n-1}(x)$ and suppose $e(1) \leqslant e(2) \leqslant \cdots \leqslant e(n)$. Then we have that $e(n) = r$.*

COROLLARY. *No monomial appearing in the* rth *term of a Baxter sequence element can appear in the* sth *term ($r \neq s$) of the same Baxter sequence element.*

LEMMA 3. *Let $x_{e(1)}x_{e(2)} \dots x_{e(n)}$ be any monomial appearing in the* rth *term of $B_1 B_2 \cdots B_{n-1}(x)$. The coefficient of $x_{e(1)} \dots x_{e(n)}$ is unity.*

Proof. This will be by induction. The lemma is obviously true for $x = (x_1, x_2, \dots)$ so assume it to be true for Baxter sequences of $(n-2)$ terms (i.e., it is true for $B_1 \cdots B_{n-2}(x)$).

Consider $B_{n-1}(B_1 \cdots B_{n-2}(x))$. By the corollary of Lemma 2 and the induction hypothesis, every monomial appearing in the rth term is different (i.e., has coefficient one). So in $xB_{n-1}(B_1 \cdots B_{n-2}(x))$, every distinct monomial appears with coefficient one.

LEMMA 4. *Suppose $x_{e(1)} \dots x_{e(n)}$ is a monomial appearing in the* rth *term of $B_1 \cdots B_{n-1}(x)$.*

Then:

$$e(k) \leqslant e(k+1) \quad \text{if} \quad B_k = P,$$
$$e(k) < e(k+1) \quad \text{if} \quad B_k = S \qquad \text{for } k = 1, 2, \dots, n-1.$$

(For example, if $x_{e(1)}x_{e(2)} \dots x_{e(5)}$ is a monomial appearing in the rth term of $PSSP(x)$ then $e(5) = r$ and $1 \leqslant e(1) \leqslant e(2) < e(3) < e(4) \leqslant e(5) = r$.)

Proof. For $k = 1, 2, \dots, n$ define an infinite sequence of indeterminates

$${}^k x = ({}^k x_1, {}^k x_2, {}^k x_3, \dots).$$

Consider a monomial ${}^1x_{e(1)}{}^2x_{e(2)} \dots {}^n x_{e(n)}$ in the rth term of

$${}^n x B_{n-1}({}^{n-1}x B_{n-2}({}^{n-2}x B_{n-3}(\dots {}^2 x B_1({}^1 x) \dots))).$$

Considering the pair $\dots {}^r x_{e(r)}{}^{r+1}x_{e(r+1)} \dots$, it is not difficult to see that

$$e(r) \leqslant e(r+1) \quad \text{if} \quad B_r = P,$$
$$e(r) < e(r+1) \quad \text{if} \quad B_r = S.$$

Now remove the upper prefixes and the proof is complete.

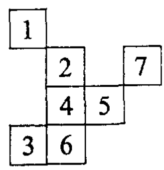
5. Main Results

5.1. *Construction of a Baxter Sequence from a Standard Numbering*

Given a frame F of n squares and ζ a standard numbering of F, define a Baxter sequence $B_1^\zeta \cdots B_{n-1}^\zeta$ constructed from ζ as follows.

Suppose $F = \{(i_r, j_r)\}_{r=1}^n$ and $\zeta(i_r, j_r) = r$ for $r = 1, 2, \ldots, n$. Put $B_r^\zeta = S$ if $i_r < i_{r+1}$, and $B_r^\zeta = P$ otherwise.

For example, construct from



the Baxter sequence $SSPPSP$.

This sequence has interpretations in other works. If the P's are replaced by $+$ signs and the S's by $-$ signs, then one produces a sequence which has appeared in works by Foulkes [2], under the name *line of route*, and Schützenberger [6]. This sequence can also be connected with the *up–down sequence* of a permutation (see [2]). For a fuller investigation into the properties and applications of Baxter sequences, the reader is referred to Thomas [7].

5.2. Theorem 1. *Let C_F be the equivalence class of members of D_F which have the index numbering ζ.*

Then $\bar{I}(C_F) = B_1^\zeta \cdots B_{n-1}^\zeta(x)$.

Proof. (i) Suppose $F = \{(i_r, j_r)\}_{r=1}^n$ where $\zeta(i_r, j_r) = r$ for $r = 1, 2, \ldots, n$. Suppose $x_{e(1)} x_{e(2)} \ldots x_{e(n)}$, $(e(1) \leqslant e(2) \leqslant \cdots \leqslant e(n))$, is a monomial occurring in $B_1^\zeta \cdots B_{n-1}^\zeta(x)$. As in Lemma 1, define a numbering η by putting $\eta(i_r, j_r) = e(r)$ for $r = 1, 2, \ldots, n$. By Lemma 1, η is uniquely determined. We now have to show that $\eta \in D_F$.

Consider (i, j) and (i', j') two squares in F. Suppose $\eta(i, j) = e(r)$, $\eta(i', j') = e(r')$.

First, suppose $i = i'$ and $j < j'$. Therefore $\zeta(i, j) < \zeta(i', j')$ and $r < r'$.

So $e(r) \leqslant e(r')$ and $\eta(i, j) \leqslant \eta(i', j')$.

Second, suppose $j = j'$ and $i < i'$. Therefore $\zeta(i, j) < \zeta(i', j')$ and $r < r'$.

Also, for some r'' such that $r \leqslant r'' \leqslant r' - 1$, $B_{r''}^\zeta = S$. So, by Lemma 4, $e(r) < e(r')$ and $\eta(i, j) < \eta(i', j')$.

Therefore $\eta \in D_F$.

Since $x_{e(1)} \ldots x_{e(n)}$ only occurs in the $e(n)$th term of $B_1^\zeta \cdots B_{n-1}^\zeta(x)$ and then only with coefficient one (from Lemmas 2 and 3) we have

$$B_1^\zeta \cdots B_{n-1}^\zeta(x) \subseteq \bar{I}(C_F).$$

(ii) Take $\eta \in D_F$ and $M(\eta) = x_{e(1)} \dots x_{e(n)}(e(1) \leqslant \cdots \leqslant e(n))$. By the construction of the index numbering, and the construction of $B_1{}^\zeta \cdots B_{n-1}^\zeta$ from the index numbering, we have that

$$B_r{}^\zeta = S \qquad \text{only if } \eta(i_r, j_r) < \eta(i_{r+1}, j_{r+1}).$$

Hence $M(\eta) \in B_1{}^\zeta \cdots B_{n-1}^\zeta(x)$ and $\bar{I}(C_F) \subseteq B_1{}^\zeta \cdots B_{n-1}^\zeta(x)$.
Therefore, by the corollary of Lemma 1,

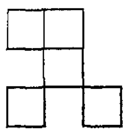
$$\bar{I}(C_F) = B_1{}^\zeta \cdots B_{n-1}^\zeta(x).$$

COROLLARY.

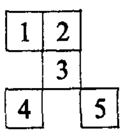
$$\bar{I}(D_F) = \sum_\zeta B_1{}^\zeta \cdots B_{n-1}^\zeta(x),$$

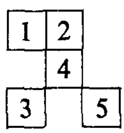
where the summation is over the standard numberings ζ of F.

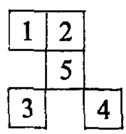
5.3. EXAMPLE.

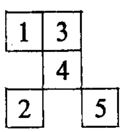
$F =$ 

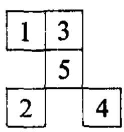
The standard numberings are

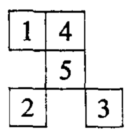
(i) 

(ii) 

(iii) 

(iv) 

(v) 

(vi) 

The corresponding Baxter sequences are

(i) $P\,S\,S\,P$ (ii) $P\,S\,P\,S$ (iii) $P\,S\,P\,P$
(iv) $S\,P\,S\,S$ (v) $S\,P\,S\,P$ (vi) $S\,P\,P\,S$

The terms of $I^4(D_F)$ (for definition, see Section 3.3) are

(i) from $P\,S\,S\,P$

$$x_1x_2x_3x_4x_4 + x_1x_1x_2x_3x_4 + x_1x_1x_2x_4x_4 + x_1x_1x_3x_4x_4 + x_2x_2x_3x_4x_4;$$

(ii) from $P\,S\,P\,S$

$$x_1x_2x_3x_3x_4 + x_1x_1x_2x_2x_4 + x_1x_1x_2x_3x_4 + x_1x_1x_3x_3x_4 + x_2x_2x_3x_3x_4;$$

(iii) from $P\,S\,P\,P$

$$x_1x_1x_4x_4x_4 + x_1x_2x_4x_4x_4 + x_1x_3x_4x_4x_4 + x_2x_2x_4x_4x_4 + x_2x_3x_4x_4x_4\,,$$
$$x_3x_3x_4x_4x_4 + x_1x_1x_3x_4x_4 + x_2x_2x_3x_4x_4 + x_1x_2x_3x_4x_4 + x_1x_1x_2x_4x_4\,,$$
$$x_1x_1x_3x_3x_4 + x_1x_2x_3x_3x_4 + x_2x_2x_3x_3x_4 + x_1x_1x_2x_3x_4 + x_1x_1x_2x_2x_4;$$

(iv) from $S\,P\,S\,S$

$$x_1x_2x_2x_3x_4;$$

(v) from $S\,P\,S\,P$

$$x_2x_3x_3x_4x_4 + x_1x_2x_2x_3x_4 + x_1x_2x_2x_4x_4 + x_1x_2x_3x_4x_4 + x_1x_3x_3x_4x_4;$$

(vi) from $S\,P\,P\,S$

$$x_2x_3x_3x_3x_4 + x_1x_2x_2x_2x_4 + x_1x_2x_2x_3x_4 + x_1x_2x_3x_3x_4 + x_1x_3x_3x_3x_4\,.$$

The corresponding equivalence classes of D_F are

(i)
$$\begin{array}{ccc} 1 & 2 & \\ & 3 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 2 & \\ 3 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 2 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 3 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 2 & 2 & \\ & 3 & \\ 4 & & 4 \end{array}$$

(ii)
$$\begin{array}{ccc} 1 & 2 & \\ & 3 & \\ 3 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 2 & \\ 2 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 3 & \\ 2 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 3 & \\ 3 & & 4 \end{array} \qquad \begin{array}{ccc} 2 & 2 & \\ & 3 & \\ 3 & & 4 \end{array}$$

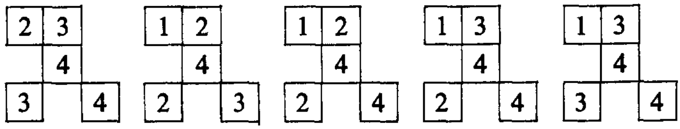
(iii)
$$\begin{array}{ccc} 1 & 1 & \\ & 4 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 2 & \\ & 4 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 3 & \\ & 4 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 2 & 2 & \\ & 4 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 2 & 3 & \\ & 4 & \\ 4 & & 4 \end{array}$$

$$\begin{array}{ccc} 3 & 3 & \\ & 4 & \\ 4 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 4 & \\ 3 & & 4 \end{array} \qquad \begin{array}{ccc} 2 & 2 & \\ & 4 & \\ 3 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 2 & \\ & 4 & \\ 3 & & 4 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 4 & \\ 2 & & 4 \end{array}$$

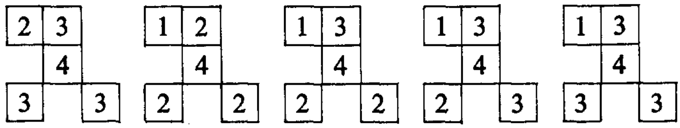
$$\begin{array}{ccc} 1 & 1 & \\ & 4 & \\ 3 & & 3 \end{array} \qquad \begin{array}{ccc} 1 & 2 & \\ & 4 & \\ 3 & & 3 \end{array} \qquad \begin{array}{ccc} 2 & 2 & \\ & 4 & \\ 3 & & 3 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 4 & \\ 2 & & 3 \end{array} \qquad \begin{array}{ccc} 1 & 1 & \\ & 4 & \\ 2 & & 2 \end{array}$$

(iv)
$$\begin{array}{ccc} 1 & 2 & \\ & 3 & \\ 2 & & 4 \end{array}$$

(v)



(vi)



5.4. *MacMahon's Work*

Some of the essentials of this situation have been observed before. MacMahon [4, p. 192–193] in a chapter on the lattice function and plane partitions examines what in effect are numberings of a frame of a partition subject to cetain row and column restrictions (in fact, the restrictions in Section 8 following). MacMahon was interested only in finding sets of inequalities that these numberings of the frame must satisfy. It is the introduction of Baxter operators that allows us actually to *generate* the complete set of solutions. MacMahon appreciated the fact that one can arrive at these inequalities by looking at the standard numberings of the frame in a way similar to that described. MacMahon, however, merely states his result without proof and as far as I am aware no published proof exists.

6. Schur Functions

Consider the Schur function corresponding to the partition (λ) of n. Suppose that it is expressed as a function of the r indeterminates $x_1, x_2, \ldots, x_r$. We shall denote this function by $\{\lambda\}^r$.

Now consider the infinite sequence

$$(\{\lambda\}^1, \{\lambda\}^2, \{\lambda\}^3, \ldots)$$

which we shall denote by $\{\lambda\}$. Then, from Theorem 1, we have

$$\{\lambda\} = P\Big(\sum_{\zeta} B_1{}^{\zeta} \cdots B_{n-1}^{\zeta}(x)\Big) = \sum_{\zeta} P(B_1{}^{\zeta} \cdots B_{n-1}^{\zeta}(x))$$

where the summation is over the standard tableaux of the partition (λ).

Alternatively, since $\{\lambda\}^r \subseteq \{\lambda\}^{r+1}$ for $r = 1, 2, \ldots$, denote by $[\lambda]$ the sequence

$$(\{\lambda\}^1, \{\lambda\}^2 - \{\lambda\}^1, \{\lambda\}^3 - \{\lambda\}^2, \ldots).$$

Then $P([\lambda]) = \{\lambda\}$ and $[\lambda] = \sum_{\zeta} B_1{}^{\zeta} \cdots B_{n-1}^{\zeta}(x)$ where the summation is over the standard numberings ζ of the frame of (λ).

7. Conjugate Frames

7.1. Definition. Given a frame $F = \{(i_r, j_r)\}_{r=1}^{n}$, define $\tilde{F}$, the *conjugate frame of* F, by $\tilde{F} = \{(j_r, i_r)\}_{r=1}^{n}$.

7.2. Theorem 2.

$$\bar{I}(D_{\tilde{F}}) = \sum_{\zeta} \tilde{B}_1^{\zeta} \cdots \tilde{B}_{n-1}^{\zeta}(x)$$

where the summation is over all standard numberings of F *and* $\tilde{B}_1^{\zeta} \cdots \tilde{B}_{n-1}^{\zeta}$ *is derived from* $B_1^{\zeta} \cdots B_{n-1}^{\zeta}$ *by*

$$\tilde{B}_r^{\zeta} = \begin{cases} P & \text{if } B_r^{\zeta} = S \\ S & \text{if } B_r^{\zeta} = P \end{cases} \qquad \text{for} \quad r = 1, 2, \dots, n-1.$$

$\tilde{B}_1^{\zeta} \cdots \tilde{B}_{n-1}^{\zeta}$ will be called the *Baxter sequence conjugate to* $B_1^{\zeta} \cdots B_{n-1}^{\zeta}$ (i.e., to find $\bar{I}(D_{\tilde{F}})$ from $\bar{I}(D_F)$, look at $\bar{I}(D_F)$ expressed as a sum of Baxter sequence elements of $(A; P, S)$ and interchange the P's and S's throughout).

7.3. *Proof of Theorem* 2. Let ζ be a standard numbering of F and suppose $F = \{(i_r, j_r)\}_{r=1}^{n}$ and $\zeta(i_r, j_r) = r$ for $r = 1, 2, \dots, n$.

Define a numbering $\tilde{\zeta}$ for $\tilde{F}$ by $\tilde{\zeta}(j_r, i_r) = r$ for $r = 1, 2, \dots, n$. This is clearly a standard numbering of $\tilde{F}$.

Therefore $\bar{I}(D_{\tilde{F}}) = \sum_{\tilde{\zeta}} B_1^{\tilde{\zeta}} \cdots B_{n-1}^{\tilde{\zeta}}(x) = \sum_{\zeta} B_1^{\tilde{\zeta}} \cdots B_{n-1}^{\tilde{\zeta}}(x)$ where the summation is over the standard numberings $\tilde{\zeta}$ of $\tilde{F}$ or alternatively, over the standard numberings ζ of F.

However, consider $B_1^{\tilde{\zeta}} \cdots B_{n-1}^{\tilde{\zeta}}$ (as defined in Section 5.1). In general, $B_1^{\tilde{\zeta}} \cdots B_{n-1}^{\tilde{\zeta}} \neq \tilde{B}_1^{\zeta} \cdots \tilde{B}_{n-1}^{\zeta}$.

7.4. *A Rectangular Condition on F*

Let us consider first, the conditions on F for $B_1^{\tilde{\zeta}} \cdots B_{n-1}^{\tilde{\zeta}}$ to equal $\tilde{B}_1^{\zeta} \cdots \tilde{B}_{n-1}^{\zeta}$ for all ζ.

If $B_r^{\zeta} \neq \tilde{B}_r^{\zeta}$, i.e., $B_r^{\tilde{\zeta}} = B_r^{\zeta}$, then either

$$i_r > i_{r+1} \qquad \text{and} \qquad j_r > j_{r+1},$$

or

$$i_r < i_{r+1} \qquad \text{and} \qquad j_r < j_{r+1}.$$

So, if for all pairs of squares (i, j), $(i', j') \in F$, either $(i, j') \in F$ or $(i', j) \in F$ (or both), then equality will hold for all standard numberings ζ of F and Theorem 2 is true immediately.

7.5. *Completion of the proof.* To prove the general case, we need to return to Section 2.3 and the definition of an index frame.

Let $F = \{(i_r, j_r)\}_{r=1}^{n}$ and consider a member η of D_F. We shall define another total ordering of the squares of F corresponding to η as follows.

(i, j) precedes (i', j') if either

$$\eta(i, j) < \eta(i', j'),$$

or

$$\eta(i, j) = \eta(i', j') \qquad \text{and} \qquad j < j'.$$

(In the case of frames satisfying the rectangular condition, this ordering is identical to that derived in Section 2.3. But it will not, in general, be the same.)

Suppose the ordering is

$$(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n).$$

Define a standard numbering of F by $\zeta(i_r, j_r) = r$ for $r = 1, 2, \ldots, n$. Using this as a definition of an index numbering, we can define an equivalence relation as in Section 2.4. (The equivalence classes, however, are not generally the same as before.)

Lemma 1 also remains true but the definition in Section 5.1 of a Baxter sequence needs altering as follows.

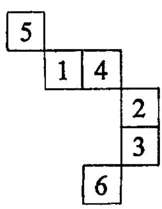
Put $B_r^{\zeta} = S$ if $j_r \geqslant j_{r+1}$, and $B_r = P$ otherwise. (Again, this is identical to Section 2.3 if F satisfies the rectangular condition.)

Now using these definitions of an index numbering and the corresponding Baxter sequence, Theorem 1 remains true.

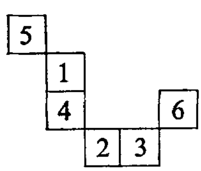
But given a standard numbering ζ of F, $B_1^{\zeta} \cdots B_{n-1}^{\zeta}$ as defined on Sections 7.3 and 5.1, is conjugate to $B_1^{\zeta} \cdots B_{n-1}^{\zeta}$ as defined in this paragraph.

Hence we have the required result.

7.6. Example.



The Baxter sequence as defined in Section 5.1 is $S\,S\,P\,P\,S$. The conjugate is



The Baxter sequence constructed from this numbering using the method in Section 7.5 is $P\,P\,S\,S\,P$.

8. Other Numberings of Frames

We could also consider the inventory of numberings of F that satisfy some other properties.

For example, suppose we are interested in numberings that are non-decreasing on both the rows and columns, i.e., $\eta(i,j) \geqslant \eta(i',j')$ whenever $i = i', j > j'$, or $j = j', i > i'$.

Given a standard numbering ζ of F, define

$$*B_1{}^{\zeta} \cdots *B_{n-1}^{\zeta} \quad \text{by} \quad *B_r{}^{\zeta} = S \quad \text{if} \quad i_r > i_{r+1}, \quad *B_r{}^{\zeta} = P \quad \text{otherwise.}$$

The required inventory is then

$$\sum_{\zeta} *B_1{}^{\zeta} \cdots *B_{n-1}^{\zeta}(x),$$

where the summation is over the standard numberings of F.

(Alternatively, put $*B_r{}^{\zeta} = S$ if $j_r > j_{r+1}$, $*B_r{}^{\zeta} = P$ otherwise. Although each standard numbering produces a different Baxter sequence, the result still holds.)

Note that Theorem 2 is not applicable in this situation. In fact, the set of numberings of F is identical to the set of numberings of $\tilde{F}$ in this situation.

The conjugate case here is the set of numberings η of F satisfying $\eta(i,j) > \eta(i',j')$ whenever $i = i', j > j'$ or $j = j', i > i'$.

The inventory of these numberings is precisely the conjugate of the last case, namely,

$$\sum_{\zeta} *\tilde{B}_1{}^{\zeta} \cdots *\tilde{B}_{n-1}^{\zeta}(x),$$

where the summation is over the standard numberings of F.

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